THE ANALYST;

OR,

MATHEMATICAL MUSEUM.

CONTAINING

NEW ELUCIDATIONS, DISCOVERIES AND IMPROVEMENTS,

IN VARIOUS BRANCHES OF THE

MATHEMATICS,

WITH COLLECTIONS OF QUESTIONS

PROPOSED AND RESOLVED

BY INGENIOUS CORRESPONDENTS.

VOL. I. No. I.

UTILE DULCI.

PHILADELPHIA:
PUBLISHED BY WILLIAM P. FARRAND AND CO.
FRY AND KAMMERER, PRINTERS.

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PREFACE.

THE Editor of the Analyst, or Mathematical Museum, has been induced to engage in the work from a thorough knowledge of its utility in spreading and improving Mathematical Science; and he indulges himself in the hope that the real friends of science in the United States are sufficiently numerous and spirited to support him in the undertaking.

It is not necessary at present to enter into a lengthy defence of the practice of publicly proposing and answering new mathematical problems. Every one who has any acquaintance with the history of mathematics knows that many valuable improvements and discoveries have resulted from the profound attention bestowed on the solution of new, curious, useful, or difficult problems. The greatest mathematicians, as Pascal, Leibnitz, The Bernoullis, Huygens, Wallis, Newton, Maclaurin, Euler, Lagrange, Emerson, Simpson, Hutton, Vince, &c. have not disdained to enter the lists, and try their strength of genius in contests of such a nature.

The truly ingenious method of differencing, *De curva in curvam** was invented by Leibnitz in attempting to resolve, in a general manner, a difficult problem proposed by James Bernoulli. James Bernoulli himself invented a profound and general method of resolving isoperimetrical problems in seeking the solution of the problem concerning the curve of swiftest descent, which was proposed by his brother John. According to Laplace and others, a vast number of improvements sprung from the prize† problems proposed by the Academy of Sciences. And to close this enume-

* A problem or two exemplifying the method *De curva in curvam*, would probably be acceptable to many of our readers: they shall be gratified as soon as convenient.

† Perhaps the most elegant and profound discovery ever produced by a prize problem was that of Maclaurin, in which he demonstrated for the first time, that, according to the known law of gravitation, an oblate speroid of uniform density would retain its figure by revolving uniformly about its less axis in a certain time, which time he determined accurately in all cases.

ration, the immortal NEWTON, whose name I am unworthy to write or to pronounce, was led to the discovery of the grand law of universal gravitation by his attempting to solve a new and curious problem proposed to him by Dr. Hook.

But let us not sit down contented with imagining that those great men who have gone before us have exhausted the subject, and left us nothing to do but to copy their writings. This were a dangerous as well as groundless idea; and can never exist in the breast of a man of real genius. Doubtless many improvements and discoveries are still treasured up to reward the ingenuity of future inquirers: may it be our lot to obtain some portion of this precious deposit.*

It is well known that various persons at different times may fall upon the same problems or solutions without any knowledge of their mutual coincidence: it will not appear surprising, therefore, if questions which have been already discussed should sometimes make their appearance as new. The assistance of the contributors in this affair is requested by the Editor; he hopes they will point out such questions already in print as coincide with those proposed from time to time as original problems. By this means we shall be enabled to learn more completely what real additions are made to the general stock of mathematical knowledge.

It will probably be expected from the present editor, by some of his fellow contributors to the Mathematical Correspondent, that he should make an apology for presuming to decide on the merits of their performances. Should such mathematicians as Craig and Maughan honour the Analyst with their contributions, he ingenuously confesses that nothing but their own consent would entitle him to the liberty of giving his judgment on their pieces.

Let it be considered, however, that he by no means pretends to dictate to the mathematicians of America;

Nec sibi regnandi veniat tam dira cupido.

In deciding on the comparative excellence of the pieces presented to him he wishes not to set up his own ideas as the stand-

* We should however be exceedingly cautious in concluding that our researches are entirely new, merely because we have not met with those of a similar kind in the common authors on mathematics: many instances of precipitancy in this respect are well known to those who are acquainted with the progress of mathematical science.

ard of taste; he will merely give his judgment according to the extent of his skill, and leave to the public the task of finally determining the comparative ingenuity of rival competitors.

It would perhaps contribute something to the progress of science, if the Editor were enabled by the sale of the work to have two Prize Questions in each number, a greater and a less. By this plan many who are not able to contend for a prize depending on certain abstruse researches might be usefully and honourably employed in resolving a prize problem of less profundity. On the other hand, mathematicians of eminence, who would not accept a prize for what cost them scarcely a thought, might find in the problems of the higher prize something worthy of attention.

It ought however to be indispensably requisite in a prize question that it may be useful in improving some important theory little known, or in discovering some new one, or, lastly, in giving some rules of practical application. It is presumed our first prize question will be found useful in one of these ways; we hope therefore it will meet with the approbation and attention of judicious mathematicians.

A comparative view of the various methods by which we arrive at the solutions of questions is at once agreeable and instructive; accordingly the Editor intends to publish as great a variety of good solutions to each question as the limits of the work will admit. This plan will undoubtedly be approved by such as duly consider its numerous advantages.

The Editor begs leave to assure the friends of science and of man, that nothing unbecoming a christian and a gentleman shall be suffered to make its appearance in the work as long as it shall be under his direction. No affected superiority shall be shown, nor contemptuous treatment of such as differ from us in opinion, or fall into errors. Let a just sense of our own imperfections teach us moderation in our judgment of others; and let us endeavour to show that we are influenced by the noblest motives, the love of elegant and useful science, and the benefit of mankind.

ROBERT ADRAIN.

THE ANALYST;

OR,

MATHEMATICAL MUSEUM.

VOLUME I. NUMBER I.

ARTICLE I.

VIEW OF THE DIOPHANTINE ALGEBRA.

Continued from Article xxvi *of the Mathematical Correspondent.*

BY ROBERT ADRAIN.

HAVING exhibited in the Mathematical Correspondent the principal elementary rules of the Diophantine Algebra, my object in the present article is to exemplify those rules in the resolution of a select number of curious problems, some of which are, I believe, entirely new.

PROBLEM I.

To find two numbers of which the sum and difference may both be squares.

SOLUTION.

Let us begin with finding such expressions for the numbers sought, that their sum may be a square. It is self-evident that if we divide any square whatever, zz into two parts, viz. u and $zz-u$ their sum $u+zz-u$ will necessarily be a square. For example, it is plain that the sum of u and $16-u$ is a perfect square. It only remains then to discover such a value for u, that the difference of u and $16-u$ may be a square, that is, we are to make $16-2u$ a rational square. Put $16-2u=nn$, and we have $u=\frac{16-nn}{2}$. If we assume $n=2$, we have $u=\frac{16-4}{2}=6$, and the other number $=16-6=10$; therefore 10 and 6 are two numbers answering the proposed problem; for their sum is 16 and their difference is 4.

To resolve the problem completely, and find general formulas expressing all the possible numbers that will answer the question, let us resume the general expressions u and $zz-u$, for the numbers required; and it is manifest we have only to make the difference $zz-2u$ a perfect square. Put $zz-2u=vv$; and we shall have $u=\frac{zz-vv}{2}$, and therefore also $zz-u=\frac{zz+vv}{2}$. The formulas therefore $\frac{zz+vv}{2}$ and $\frac{zz-vv}{2}$ are general expressions for all the possible numbers fulfilling the conditions of the question. If $z=2$, $v=1$, then the required numbers are $\frac{5}{2}$ and $\frac{3}{2}$.

ANOTHER SOLUTION.

The principle of the preceding solution consisted in first determining such a relation between the unknown quantities as may fulfil one of the required conditions, and afterwards in adding a new relation between the same unknown quantities by means of which the remaining condition may also be fulfilled: and this manner of procedure is more extensively useful in the Diophantine Algebra than any other method hitherto discovered. There is however another method which is frequently of considerable advantage; it consists in determining the unknown quantities from such equations as fulfil all the conditions of the question at the same time. The spirit of this method will be seen in the following solution to the preceding problem.

Let u and y be the required numbers; and aa and bb the two squares to which the sum and difference of u and y are to be equal; and by the question we have the equation $u+y=aa$, and $u-y=bb$. Now resolving these equations with respect to u and y, we obtain $u=\frac{aa+bb}{2}$ and $y=\frac{aa-bb}{2}$; in which expressions we may choose a and b at pleasure. If we suppose $aa=25$ and $bb=9$, we find $u=17$ and $y=8$.

This problem is resolved in a very different manner in Emerson's Algebra, Book II. Prob. lxxi. But that excellent author does not seem to have remarked that the problem is only of the first or simplest degree; as he has without necessity applied to it a method for the second degree.

It will readily occur to the attentive reader, that if instead of requiring two numbers of which the sum and difference may be *squares*, we had demanded two numbers of which the sum and difference might be *cubes*, or *fourth* powers or n^{th} powers; or the sum an m^{th} power, and the difference an n^{th} power, we might have resolved the problem in the same manner as above.

For example. *To find two such numbers that their sum may be a square, and their difference a cube.*

Assume $u+y=a^2$, and $u-y=b^3$; by addition, $2u=a^2+b^3$: by subtraction, $2y=a^2+b^3$. Whence $u=\frac{a^2+b^3}{2}$, and $y=\frac{a^2-b^3}{2}$.

If $a^2=16=4^2$ and $b^3=8=2^3$ then $u=\ 2$ and $y=4$.

PROBLEM II.

To find two such numbers that each added to twice the other may be a rational square.

SOLUTION.

Let u and y be the required numbers: aa and bb the squares mentioned in the question, then we have by the question $u+2y=aa$, and $2u+y=bb$.

By addition $3u+3y=aa+bb$, whence $u+y=\frac{aa+bb}{3}$; this last equation taken from each of the preceding gives $u=\frac{2bb-aa}{3}$ and $y=\frac{2aa-bb}{3}$; and a and b may be taken at pleasure. If $a=2$ and $b=1$, we have $u=-\frac{2}{3}$ and $y=\frac{7}{3}$. If $a=12$ and $b=9$, we have $u=6$, and $y=69$: now $69+6\times2=69+12=81=9^2$, and $6+69\times2=138+6=144=12^2$.

This problem is only a particular case of the following. To find u and y such that $au+by$ and $a'u+b'y$ may be rational squares, a, b, a' and b' being any given numbers.

To resolve this problem we have only to assume $au+by=mm$, and $a'u+by'=nn$: and finding the values of u and y by the common methods for simple equations we find $u=\frac{b'mm-nnb}{ab'-a'b}$, and $y=\frac{ann-a'mm}{ab'-a'b}$.

It may not be improper on this occasion to give a specimen of a general method of resolving simple equations, which is not I presume very commonly known among the readers of the Analyst.

I. To resolve the equations $au+by=m$, $a'u+b'y=n$. Multiply the former equation by c, and we have $a\,c\,u+b\,c\,y=c\,m$; to this add the second equation, and we have $a\,c\,u+a'\,u+b\,c\,y+b'\,y=cm+n$; that is $(a\,c+a')\times u+(b\,c+b')\times y=c\,m+n$: now to remove y put its coefficient $b\,c+b'=o$, whence $c=-\frac{b'}{b}$, and since $(a\,c+a')\,u=cm+n$, we have $u=\frac{cm+n}{ac+a}$. If we make the coefficient of u, viz. $a\,c+a'=o$, we have $c=-\frac{a'}{a}$, and $y=\frac{cm+n}{bc+c'}$.

II. Given $\left\{\begin{array}{l} a\ u+b\ y+c\ z=m \\ a'\ u+b'\ y+c'\ z=m' \\ a''u+b''y+c''z=m'' \end{array}\right\}$ to find u, y and z. Multiply the first and second equations by d and e respectively, and to the sum of the products add the third equation, and we have $(a\ d+a'e+a'')\ u+\ (b\ d+b'e+b'')\ y+\ (c\ d+c'e+c'')\ z=dm+em'+m''$. Let us now make the coefficients of y and z each $=o$, and we have the three equations $b\ d+b'e+b''=o$, $c\ d+c'e+c''=o$, $(a\ d+a'e+a'')\ u=d\ m+e\ m'+m''$. From the first two of these equations we obtain d and e by the method of the preceding example, and from the last we have $u=\frac{dm+em'+m''}{da+e\ a'+a''}$.

We have just shown how to make $au+by$ and $au'+b'y$ rational squares; and it may be remarked that the same method may be applied if the formulas $au+by$ and $a'u+b'y$ are to be made cubes, biquadrates, &c. or more generally, if the former must be an m^{th} power, and the latter an n^{th} power, we have only to resolve the simple equations $au+by=A^m$, and $a'u+b'y=B^n$.

If three or more numbers u, y, z, &c. are required, so that $a\ u+b\ y+c\ z$ may be an m^{th} power, $a'u+b'y+c'z$ an n^{th} power, and $a''u+b''y+c''z$ an r^{th} power, there being as many equations as unknown quantities: it is evident we have only to resolve the simple equations $a\ u+b\ y+c\ z=A^m$, &c.

PROBLEM III.

To find two numbers such that each added to the square of the other may be a perfect square.

SOLUTION.

Let u and y represent the numbers sought: and the problem is to render the two formulas $uu+y$ and $yy+u$ rational squares.

To effect this, let us begin with the formula $uu+y$. Suppose $uu+y=pp$, whence $y=pp-uu$, which expression for y will evidently render $uu+y$ a square, whatever values u and p have.

The second formula $yy+u$ will become by substitution $(pp-uu)^2-u=u^4-2p^2u^2+u+p^4$, which must also be a square. The celebrated Euler, in resolving this problem, having arrived at the formula $u^4-2p^2u^2+u+p^4$ nearly as above, abandons it; because, says he, it would be difficult to resolve.

It is certainly a curious circumstance that the most expert and sagacious analyst of the eighteenth century should have found a difficulty in rendering $u^4+2p^2u^2+u+p^4$ a rational square. We have only to assume $u=4ppuu$, and $u^4-2ppuu+u+p^4$ becomes $u^4+2ppuu+p^4=(uu+pp)^2$, which is manifestly a complete square. From $u=4ppuu$ we obtain $u=\frac{1}{4p^2}$, and therefore $y=pp-uu=$

$pp-\frac{1}{16p^4}$. We may now assume any number at pleasure for p, and we shall obtain numbers answering the question. Suppose $p=1$, then $u=\frac{1}{4}$ and $y=\frac{15}{16}$, which fulfil the required conditions; for $uu+y=\frac{1}{16}+\frac{15}{16}+\frac{16}{16}=1=(1)^2$; again, $y^2+u=\frac{225}{256}+\frac{1}{4}=\frac{225}{256}+\frac{64}{256}=\frac{289}{256}=(\frac{17}{16})^2$.

The formula $(uu-pp)^2+u$ may also be easily made a square in various other ways. If we throw it into the form $(u+p)^2\times(u-p)^2+u$, and put $p+u=v$, we shall have by substituting for p, $(u+p)^2\times(u-p)^2+u=vv(v-2u)^2+u=v^4-4v^3u+4uuvv+u=v^4+4v^3u+4uuvv+u-8v^3u$: now this last is evidently a square when $u-8v^3u=o$, for it then becomes $(vv+vu)^2$. From the equation $u-8v^3u=o$ we obtain $8v^3=1$, and $v=\frac{1}{2}$, whence $p=\frac{1}{2}-u$, and $pp-uu=-u+\frac{1}{4}$; therefore $y=-u+\frac{1}{4}$, or $u+y=\frac{1}{4}$: from which we deduce this remarkable rule: If the fraction $\frac{1}{4}$ be divided into any two parts whatever, those parts will answer the proposed problem. Suppose we divide $\frac{1}{4}=\frac{3}{12}$ into $\frac{1}{12}$ and $\frac{1}{6}$; we have $(\frac{1}{6})^2+\frac{1}{12}=\frac{1}{36}+\frac{1}{12}=\frac{4}{36}=(\frac{1}{3})^2$, and $(\frac{1}{12})^2+\frac{1}{6}=\frac{1}{144}+\frac{1}{6}=\frac{1}{144}+\frac{24}{144}=\frac{25}{144}=(\frac{5}{12})^2$.

ANOTHER SOLUTION.

We have in the preceding solution fulfilled the required conditions of the question successively: we proceed now to resolve the problem by fulfilling both the conditions together. Since $uu+y$ and $yy+u$ are to be squares, let $u+a$ and $y+b$ be the roots of those squares, and we have the equations $uu+y=uu+2au+aa$, $yy+u=yy+2by+bb$; whence $y=2au+aa$, and $u=2by+bb$. These last two equations resolved by the known rules for simple equations give $u=\frac{2baa+bb}{1-4ab}$, and $y=\frac{2abb+aa}{1-4ab}$; in which a and b may be taken at pleasure; but in order that u and y may be both positive, attention must be paid to the assumed values of a and b. Those formulas express all the possible values of u and y that can answer the question, as is evident from the method of investigation; for whatever be the root of the square $uu+y$, it is manifest that it must be contained in the formula $u+a$, in which a may be taken at pleasure.

By adding the values of u and y we have $u+y=\frac{aa+bb+2ab(a+b)}{1-4ab}$, which equation by assuming $a+b=\frac{1}{2}$, becomes $u+y=\frac{1}{4}$, as in the foregoing solution.

PROBLEM IV.

To find three numbers such that the square of each increased by the sum of the other two may be a square.

SOLUTION.

Let u, y, z be the numbers sought; we are therefore to assign such values for u, y and z, that $uu+y+z$, $yy+u+z$, $zz+u+y$ may be rational squares.

Assume $m-u$, $n-y$, $r-z$ for the three roots of these formulas; and we have the three equations,

$$uu+y+z=uu-2mu+mm,$$
$$yy+u+z=yy-2ny+nn,$$
$$zz+u+y=zz-2rz+rr:$$

which by taking the equal squares from both sides become $y+z=-2mu+mm$, $u+z=-2ny+nn$, $u+y=-2rz+rr$; and by transposition we have $y+z+2mu=mm$, $u+z+2ny=nn$, $u+y+2rz=rr$.

These three equations resolved by the rules for simple equations give the numbers sought in general terms expressing all the possible answers.

If we suppose m, n and r equal to 2, 3 and 4, respectively, we obtain $u=\frac{45}{176}$, $y=\frac{203}{176}$, $z=\frac{321}{176}$.

It would not be difficult to try if these numbers really answered the question.

The method of resolution here given is applicable to many other questions. For example, let u, y, z and v be required such that

$$uu+a\ y+b\ z+c\ v$$
$$yy+a'\ u+b'\ z+c'\ v$$
$$zz+a''\ u+b''\ y+c''\ v$$
$$vv+a'''u+b'''y+c'''z$$

may all be rational squares: a, a', a'', a''', b, b', &c. being any given numbers positive or negative. By assuming $m-u$, $n-y$, $r-z$, $s-v$, for the roots, we immediately reduce the business to the resolution of common simple equations.

As another example: It is required to find three squares in arithmetical progression.

It is plain that $uu-y$, uu, $uu+y$ may represent any three numbers in arithmetical progression, the mean being a square: we have therefore to make $uu-y$ and $uu+y$ squares.

Assume $u-m$ and $u+n$ for the roots; and we have the equations $uu-y=uu-2mu+mm$, $uu+y=uu+2nu+nn$, whence $2mu-y=mm$ and $-2nu+y=nn$; and therefore by simple equations, $u=\frac{1}{2}\times\dfrac{mm+nn}{m-n}$, and $y=mn\times\dfrac{m+n}{m-n}$.

If we assume $m=2$, $n=1$, we have $u=\frac{5}{2}$ and $y=6$, whence $uu-y=\frac{1}{4}$, $uu=\frac{25}{4}$, $uu+y=\frac{49}{4}$; or rejecting the common denomi-

nator we have the three squares 1, 25, 49, in arithmetical progression.

Three squares in arithmetical progression may be expressed in general by

$$(mm+2mn-nn)^2,$$
$$(mm+nn)^2,$$
$$(nn+2mn-mm)^2.$$

All the problems we have hitherto resolved in the present paper, together with a multitude of others, are comprehended in the following very general expressions.

Let u, y, z, v, &c. be any number of unknown, and a, b, c, d, e, f, &c. known quantities: we are to make rational squares of such formulas as

$$(a\,u+b\,y+c\,z+d\,v+e\,)^2+fu+gy+hz+iv+k$$
$$(a'u+b'y+c'z+d'v+e')^2+f'u, \text{ \&c.}$$
$$\text{\&c.} \qquad \text{\&c.}$$

there being as many similar formulas as unknown quantities.

The method of solution consists in giving such forms to the roots of the squares, that the resulting equations may all be simple. Thus, for the root of $(au+by+cz, \text{ \&c.})^2+$, &c. we are to assume $(au+by+cz+, \text{ \&c.})+m$; in like manner for the root of $(a'u+b'y, \text{ \&c.})^2+f'u$, &c. we are to assume $(a'u+b'y, \text{ \&c.})+n$. By means of this general method we may readily obtain the answers to many problems which scarcely admit of solutions by any other method.

PROBLEM V.

To investigate all the possible square numbers such that the sum of two of them may be a square.

SOLUTION.

Let uu and yy be any two square numbers of which the sum $uu+yy$ is a square.

The square root of $uu+yy$ being greater than u may be universally expressed by $u+ay$. Let us therefore assume $uu+yy=(u+ay)^2=uu+2auy+aayy$, and therefore $yy=2auy+aayy$, whence by division $y=2au+aay$, and therefore $y=\frac{2au}{1-aa}$, or $\frac{y}{u}=\frac{2a}{1-aa}$.

As a may be fractional, put it $=\frac{n}{m}$, and we have $\frac{y}{u}=\frac{\frac{2m}{n}}{1-\frac{nn}{mm}}=\frac{2mn}{mm-nn}$;

therefore, universally, $u:y::mm-nn:2mn$.

It is manifest therefore that $a(mm-nn)$ and $2amn$ must contain all rational numbers, such that the sum of the squares of two

of them may be a square; of course the three sides of every right angled triangle when rational, are contained in the formulas $a(mm+nn)$, $a(mm-nn)$ and $2amn$.

Assume $a=1$, $m=2$, $n=1$, and the values of these three expressions become 5, 4, and 3: and we have evidently $3^2+4^2=5^2$.

These few examples may be sufficient to give the learner some idea of the methods to be pursued in the solution of Diophantine problems. We should willingly give a more enlarged specimen of this doctrine, were we not afraid of fatiguing such of our readers as are not profoundly skilled in analysis. But in order to make some amends for our brevity to those who find pleasure in such curious speculations, we shall subjoin a few questions for exercise, which we presume are entirely new: and this we do the more readily, because researches of this nature are well calculated to give the student that sagacity and address in the management of algebra, which constitute at once its elegance and its utility.

PROBLEM I.

To find two numbers such that the sum of their squares may be a cube, and the sum of their cubes a square.

PROBLEM II.

To find two integers such that the sum of their cubes increased by their product may be a square.

PROBLEM III.

To find two integers such that the sum of their cubes increased by their product may be a cube.

PROBLEM IV.

To find two numbers such that their sum is equal to the difference of their biquadrates.

PROBLEM V.

To find two numbers such that their difference is equal to the difference of their biquadrates.

PROBLEM VI.

To find the three sides of a rational right angled plane triangle such that the square of each leg increased by the biquadrate of the other may be a square.

PROBLEM VII.

To find two numbers such that their sum may be a square, the sum of their squares a square, the sum of their cubes a square, and the sum of their biquadrates a cube.

PROBLEM VIII.

To find three square numbers such that each increased by the square of their sum may be a square.

PROBLEM IX.

To find three or more numbers such that when each is subtracted from the sum of their squares the remainders may be squares.

PROBLEM X.

To find four numbers such that if each be added to and subtracted from the square of their sum, the sums and remainders may be all squares.

PROBLEM XI.

To find four integers such that the sum of every two may be a square.

PROBLEM XII.

To find four or more cubes of which the sum may be a cube.

ARTICLE II.

OBSERVATIONS ON THE STUDY OF MATHEMATICS.

BY THE EDITOR.

THE advantages which might be derived from the study of mathematics are generally lost in a great degree by young persons, in consequence of the wrong methods they pursue in the commencement of their course. It is no unusual practice for many to enter upon the practical branches, such as surveying mensuration, gauging, navigation, geography, &c. without any previous knowledge of geometry, and frequently without even a moderate skill in common arithmetic: and after having toiled laboriously through the most common parts of these sciences without pleasure and without improvement, they are finally compelled by insurmountable obstacles to give up their studies, in the full persuasion that mathematics are the least valuable, and at the same time the most insipid of all human inquiries. This erroneous conclusion may be pardoned in persons who have attempted the business on a mistaken plan; and who are therefore unable to form a correct judgment on the subject.

Such however as wish to make any proficiency in mathematical science, and to enjoy the real advantages which may be derived from it, must follow a very different path. There is in fact but one method by which we can render the study of mathematics valu-

able; I mean, by laying in the first place a good foundation in the elements of arithmetic, geometry, and algebra. Without a competent knowledge of these we only deceive ourselves in hoping to obtain an accurate acquaintance with the practical branches, or with the various departments of mathematical philosophy; and with the assistance of these principles we shall meet few difficulties which we shall not be able to surmount.

It were easy to point out multitudes of cases, in which a knowledge of the fundamental principles of those sciences might be applied with advantage to practical purposes: it may not be impertinent to give the following example. It will readily be allowed, that it is a useful problem in geography to determine the distance between two places on the surface of the earth.

By the help of a globe we may indeed resolve the problem with the greatest facility: but a globe is not always at hand; and unless the two places can be found on the same map at no great distance from each other, the problem is, generally speaking, beyond the reach even of those who have gone through a course of practical geography. Yet this problem may be resolved with great ease and expedition by scale and compass, provided that one has a tolerable notion of the first principles of geometry: for though at first sight the problem seems to belong (and in fact it does) to spherical trigonometry; yet it may be very easily resolved by plane geometry without any knowledge of spherics.

Imagine the two places to lie in the circumference of the base of a hemisphere; and supposing two meridians to pass from those places to one of the poles of the earth, which will be somewhere on the surface of that hemisphere, we shall have given in a spherical triangle the two distances from the places to the pole, which are the complements of their latitudes, and the contained angle which is their difference of longitude, to find the distance between the places, which is the base of the triangle. From the centre of the sphere imagine two straight lines to be drawn through the two places in the plane of the base of the hemisphere, and produced in this plane until they become the secants of the adjacent sides or polar distances; and from the vertex of the triangle, or point on the surface of the hemisphere which is the pole of the earth, let there be drawn two straight lines to the extremities of the forementioned secants, and these last drawn lines will evidently be the tangents of their respective sides, and the angle contained by those tangents will be the difference of longitude of the places, as the angle contained by the two secants shows the degrees in the base of the triangle which is the distance of the places.

Now because the polar distances are known, their tangents T, t, and secants S, s, are also known, and the angle contained between T and t, to find the angle contained between S and s. But this is

a problem of a very simple nature, and easily performed by plane geometry.

We have given the four sides T, t, S, s, of a plane trapezium, and the angle contained between T and t, to construct the trapezium and measure the angle contained between S and s. The values of T, t, S, s, may either be taken off the common scale of tangents and secants, or may be easily found by construction to any radius at pleasure.

If the polar distances or latitudes of the places were given, and their distance, to find the difference of longitude, we might proceed on the same principles: there will in this case be given the four sides, S, s, T, t, of a trapezium, and the angle contained by S and s, to find that contained between T and t, which will be the difference of longitude.

The same constructions may evidently be applied to many other similar problems; we may even find by this method the angles at the places made by the distance and the meridians, which are commonly called angles of position, or the bearings of the places from each other: But not to enter into minutiæ at present, it will be sufficient to remark that our constructions furnish practical solutions to two or rather three cases of spherics.

I. When two sides and the contained angle are given, to find the base and other angles.

II. When the three sides are given to find the three angles.

III. When the three angles are given to find the three sides.

In this last case we must take the measures of the supplements of the three given angles for the three sides of a new triangle, and having found by the preceding construction the three angles of this new triangle, the measures of their supplements will be the required sides of the proposed triangle.

In general, the practical branches of mathematics, and the several departments of mathematical philosophy borrow one or more principles or facts from observation and experience; and these fundamental principles being once known and established, the several branches of practical mathematics become so many researches in the abstract sciences of arithmetic, geometry and algebra. Practical surveying is founded on the supposition, that notwithstanding the rotundity of the earth, such portions of its surface as are usually included in surveys may be taken for planes without any sensible error: and since by the compass and chain we can find the courses and distances round a tract of land, the whole art is resolved into one general problem in pure geometry, viz.

Given the sides and angles of any plane rectilineal figure to find its area.

The importance of this problem has made it an object of attention among geometricians in all ages, and at length they have dis-

covered a general, accurate, and easy method of solution which is scarcely susceptible of farther improvement.

Navigation requires a much more profound knowledge of mathematics than surveying, because the portions of the earth's surface which enter into this research cannot be considered as planes; we must therefore have recourse to the geometry of curve lines and curve surfaces. The general principles on which navigation is founded are the following: 1st, The true figure and dimensions of the earth. 2d, The general property of the magnetic needle by which it is possible to make a ship describe on the surface of the sea, a line that intersects all the meridians in any proposed constant angle. 3d, The practicability of measuring the distance sailed by the ship on this line by means of an instrument called the log. These principles and facts being once admitted, the art of navigation (exclusive of that part which requires the assistance of astronomical observations) is reduced to a speculation in pure geometry and algebra.

For example, there being given the latitudes and longitudes of two places, it is required to find the course and distance between them. This question when reduced to pure geometry, may be expressed thus: Given the positions of two points on the surface of a globe of a given magnitude which is supposed to have poles and meridians; it is required to find the length of a line lying between these points, and intersecting all the intermediate meridians in one and the same constant angle, and to determine the magnitude of this angle.

Again, there may be given the latitude and longitude of the place from which a ship departs, with the course and distance made good; to find the latitude and longitude of the ship. This is also a problem in pure geometry, viz. Given the position of a point on the surface of a globe of given magnitude, which is supposed to have poles and meridians, and the length of a line extending from the given point, and intersecting all the meridians at one and the same given angle; to determine the position of the extremity of this line, that is, its distance from the pole, and the angle contained between two meridians passing through its beginning and its end.

The solution of these grand problems can be accurately understood by none but such as have a considerable knowledge in algebra and geometry; and the difficulty of the investigation is still farther increased if we take into consideration the spheroidal figure of the earth. It is principally to the learned geometricians of the seventeenth century that mankind are indebted for the true principles of this important science. Millions are every day enjoying the advantages resulting from the geometrical labours of *Mercator*, *Wright*, *Halley*, *Newton*, while they remain ignorant of the names of their benefactors, and perhaps consider the study of algebra and geometry as an unprofitable waste of talents and of time.

ARTICLE III.

NEW QUESTIONS,

TO BE ANSWERED IN THE NEXT NUMBER.

QUESTION I.

By William Cherington, Reading.

A and *B* having entered into play, *B* was the winner. I own, says *A*, I have lost, but I forget how much our stakes were; to which *B* replied, our stakes did not amount to nine pounds, but if you add four pounds to their treble, the sum will then exceed nine pounds by double the sum they now fall short of it; hence you are desired to show the stakes of those two ill-employed gentlemen.

QUESTION II.

By an old Soldier.

A general putting his army through a variety of evolutions, disposed them at first in the form of an exact square, he afterwards threw them into a rectangle of which one side contained fifty eight men more than the other: but before we give an account of his other manœuvres, let us know if you please how many men were under his command.

QUESTION III.

By John Capp, Harrisburg.

A gentleman lent out a thousand dollars at six per cent. per annum simple interest, both principal and interest being payable at the expiration of every day; now it is required to find for what sum the lender may draw upon the borrower at the expiration of every day, that the principal with the interest may last him just 365 day.

QUESTION IV.

By the same.

Harrisburgh being just one hundred miles from Philadelphia, a traveller at the latter starts for the former at the rate of four miles per hour, and travels with a velocity always proportional to the square root of his distance from Harrisburg; now the question is, how long will the traveller be in arriving at Reading, which lies on his road at the distance of fifty-six miles from Philadelphia?

QUESTION V.

By Ebenezer R. White, Danbury, Connecticut.

In all right-angled triangles, the sum of the hypothenuse and one leg divided by twice the hypothenuse gives the square of the cosine of half the included angle: a demonstration is required.

QUESTION VI.

By the same.

In all oblique-angled triangles, if from the square of the sum of the two sides including any angle you subtract the square of the other side, and divide the remainder by four times the product of the first-mentioned sides, you will have the square of the cosine of half the included angle: a demonstration is required.

QUESTION VII.

By James M'Ginnis, Harrisburg.

Given the area of a plane rectilineal triangle $=14.749=a$, and if a perpendicular be let fall from the vertical angle on the base, the rectangle of the segments of the base multiplied by the greater segment $=47.014176=b$, and the difference of the segments multiplied by the less segment $=4.1712=c$; to determine the triangle.

QUESTION VIII.

By John Gummere, near Burlington, New-Jersey.

In a right-angled triangle ABC having the right angle at B, are given the perpendicular $BC=120$, the angle A at the base $=32°\ 40'$, and the angle $CAD=17°\ 10'$, contained between the hypothenuse AC and a straight line AD meeting the perpendicular in D; to determine the length of the straight line CE meeting the base AB and line AD in E and F, when the rectangle $AE \cdot EC$ is equal to the rectangle $AF \cdot FC$.

IX. PRIZE QUESTION.

By the Editor.

Among the many precious antiquities destroyed by the Caliph Omar in the city of Alexandria was a magnificent temple dedicated to geometry, and stored with all the treasures of ancient science. The edifice consisted of a cylindrical tower one hundred feet in diameter and as many in height, with a roof constructed in the following manner: the ridge of the roof was a straight line directly over a diameter of the upper base of the tower, to which it was both equal and parallel, at the distance of fifty feet; and the rafters extending from all points of the ridge to the circular eaves of the upper base were straight lines, at right angles to the ridge. As none were permitted to enter this venerable temple of science but such as could determine at least the solid content of the cavity of the roof, if not its superficies; the problem is here proposed to the geometers of America, and he who gives the best investigation of both surface and solidity shall carry off the prize.

RULES TO BE OBSERVED BY CONTRIBUTORS.

1. All communications must be post paid and directed to Robert Adrain, Editor of the Analyst, Reading, Pennsylvania.

2. Those who wish to have new questions inserted must send a true solution along with each question.

3. Any mathematical question sent to the Editor, with or without a solution, and accompanied by a prize of not less than six dollars in value, shall, if judged admissible, be published as a prize question, with the proposer's name; and the prize shall be awarded, by the Editor, to the author of the best satisfactory solution. But if no such solution shall be furnished within five months after publication, the prize may then be appropriated, by the donor or the Editor, to some other suitable question.

TO SUBSCRIBERS.

The publication of the two first numbers of the Mathematical Museum has been from necessity delayed some time in order to procure the necessary characters from the type foundery. The succeeding numbers may be expected more punctually.